Phase 13A – Mathematical Consolidation  
Part 2: ψ Invariants and Conservation Laws via Noether Analysis

Goal  
I derive invariants and conservation laws of ψ-gravity using Noether’s theorem, correcting the prototype so that the Laplacian is truly covariant and the Lagrangian admits well-defined currents. I keep the core relation as the defining link to the force .

Canonical Objects (from Part 1)

Effective potential:

Plain-text:  
Φ(x) = S(x) + g^{μν} J\_μ(x) J\_ν(x)

Covariant Laplacian:

Plain-text:  
∇²Φ = g^{μν} ∇\_μ ∇\_ν Φ

Gravity density:

Plain-text:  
G(x) = (∇²Φ) ψ(x)

Force:

Plain-text:  
F\_μ(x) = −∇\_μ G(x)

Action Principle (with first-derivative form)  
To avoid higher-derivative subtleties in Noether analysis, I integrate by parts the coupling and work with a dynamically equivalent Lagrangian density (up to a boundary term). I allow a ψ-kinetic prefactor to encode medium effects while preserving the core equation.

Original:

Plain-text:  
L\_orig = ½ K(Φ) g^{μν} ∇\_μψ ∇\_νψ + ½ (∇²Φ) ψ² − V(ψ, Φ)

First-derivative equivalent (discarding a total divergence):

Plain-text:  
L = ½ K(Φ) g^{μν} ∇\_μψ ∇\_νψ − ½ g^{μν} (∇\_μΦ) ∇\_ν(ψ²) − V(ψ, Φ)

Equivalence identity used:

Plain-text:  
∫ √(-g) ψ² ∇²Φ = −∫ √(-g) g^{μν} (∇\_μΦ) ∇\_ν(ψ²) (up to boundary)

Symmetry Classes and Conditions

**Internal ψ-scaling (global rescaling)**  
If the action is invariant under (infinitesimal: ) with , then a conserved current follows. A sufficient condition is that all ψ-dependences appear as (or more generally with a uniform scaling weight) and depends only on Φ, not on ψ.

**Translations / Poincaré invariance**  
If is Minkowski and background fields respect Poincaré symmetry (or more generally, if the background admits Killing vectors), then energy-momentum is conserved.

**Harmonic gauge freedom of Φ**  
Because only enters the defining coupling,

leaves the dynamics generated by unchanged (a redundancy rather than a standard Noether symmetry).

Plain-text:  
Φ → Φ + h, with ∇²h = 0, leaves L\_orig unchanged (up to boundary).

**Lorentz rotations of the current**   
Since is a scalar, the construction is Lorentz covariant; conserved angular momentum currents exist in Minkowski backgrounds.

Noether Currents

**(A) ψ-scaling current (internal, global)**  
With , , the Noether current is

Plain-text:  
J^μ\_(ψ-scale) = K(Φ) g^{μν} (∇\_νψ) ψ

Conservation (on-shell) requires the action to be invariant under ψ-scaling (e.g., or more generally homogeneous of degree 2 in ψ while is ψ-independent):

Plain-text:  
∇\_μ J^μ(ψ-scale) = 0 (on equations of motion and when the symmetry holds)

**(B) Stress-energy tensor and translation invariance**  
A symmetric energy-momentum tensor (Hilbert or improved canonical) yields:

Plain-text:  
∇\_μ T^{μν} = 0 (in symmetric backgrounds)

A convenient form (from the first-derivative L) is

where .

Plain-text:  
T^{μν} = K(Φ) ∇^μψ ∇^νψ − ½ g^{μν} K(Φ) (∇ψ)² − ½[ ∇^μΦ ∇^ν(ψ²) + ∇^νΦ ∇^μ(ψ²) ] + ½ g^{μν} (∇\_αΦ) ∇^α(ψ²) − g^{μν} V(ψ, Φ)

**(C) Angular momentum (in Minkowski)**  
With , one has

Plain-text:  
∂\_λ M^{λμν} = 0 (flat spacetime)

ψ Invariants (illustrative, background-covariant)

**ψ norm**

Plain-text:  
I\_ψ = ∫\_Σt d³x √γ ψ²

**Current-weighted ψ norm**

Plain-text:  
I\_{Jψ} = ∫\_Σt d³x √γ ψ² (J\_μ J^μ)

**Laplacian-coupled functional (harmonic-shift invariant)**

Plain-text:  
I\_{Φψ} = ∫ √(-g) ψ² ∇²Φ = −∫ √(-g) g^{μν} (∇\_μΦ) ∇\_ν(ψ²)

**Conserved charge (ψ-scaling, when symmetry holds)**

Plain-text:  
Q\_ψ = ∫Σt d³x √γ n\_μ J^μ(ψ-scale)

Here is the unit normal to the slice , the induced metric determinant.

Conditions for Exact Conservation

* **ψ-scaling:** independent of ψ and homogeneous of degree 2 in ψ (e.g., ); then on-shell.
* **Translations:** Background must admit the corresponding Killing vectors (e.g., Minkowski) for .
* **Harmonic shift of Φ:** with leaves invariant (boundary-insensitive).

Python Symbolic Prototype (corrected Laplacian and currents)

# simulations/phase13A\_part2\_noether\_invariants.py  
import sympy as sp  
  
# Coordinates and flat metric (Minkowski signature)  
t, x, y, z = sp.symbols('t x y z')  
coords = (t, x, y, z)  
eta = sp.diag(-1, 1, 1, 1) # g^{μν}  
  
# Fields  
psi = sp.Function('psi')(\*coords) # ψ(x)  
S = sp.Function('S')(\*coords) # space(x)  
J = [sp.Function(f'J{mu}')(\*coords) for mu in range(4)] # current components  
  
# Current squared J^2 = g^{μν} J\_μ J\_ν  
current\_sq = sum(eta[mu,nu]\*J[mu]\*J[nu] for mu in range(4) for nu in range(4))  
  
# Effective potential Φ = S + J^2  
Phi = S + current\_sq  
  
# Covariant Laplacian in flat space: ∇²Φ = g^{μν} ∂\_μ∂\_ν Φ  
laplacian\_Phi = sum(eta[mu,nu]\*sp.diff(Phi, coords[mu], coords[nu]) for mu in range(4) for nu in range(4))  
  
# K(Φ): kinetic prefactor (medium response); keep symbolic  
K = sp.Function('K')  
KPhi = K(Phi)  
  
# First-derivative equivalent Lagrangian density:  
# L = 1/2 K(Φ) g^{μν} ∂\_μψ ∂\_νψ − 1/2 g^{μν} (∂\_μΦ) ∂\_ν(ψ^2) − V(ψ, Φ)  
V = sp.Function('V')(psi, Phi)  
  
grad\_psi = [sp.diff(psi, c) for c in coords]  
grad\_Phi = [sp.diff(Phi, c) for c in coords]  
grad\_psi\_sq = sum(eta[mu,nu]\*grad\_psi[mu]\*grad\_psi[nu] for mu in range(4) for nu in range(4))  
  
# ∂\_ν(ψ^2)  
grad\_psi2 = [sp.diff(psi\*\*2, c) for c in coords]  
coupling\_first\_deriv = sum(eta[mu,nu]\*grad\_Phi[mu]\*grad\_psi2[nu] for mu in range(4) for nu in range(4))  
  
L = sp.Rational(1,2)\*KPhi\*grad\_psi\_sq - sp.Rational(1,2)\*coupling\_first\_deriv - V  
  
# Noether current for ψ-scaling: J^μ = (∂L/∂(∂\_μψ)) \* ψ  
dL\_d\_dpsi = []  
for mu in range(4):  
 # derivative of the kinetic term wrt ∂\_μ ψ  
 term\_kin = sum(eta[mu,nu]\*KPhi\*grad\_psi[nu] for nu in range(4))  
 # contribution from the coupling term: since ∂\_ν(ψ^2) = 2 ψ ∂\_ν ψ,  
 # the derivative of -1/2 g^{αβ} (∂\_α Φ) ∂\_β(ψ^2) w.r.t. ∂\_μ ψ gives:  
 # -(1/2) \* g^{αβ} (∂\_α Φ) \* 2 ψ \* δ^μ\_β = - ψ \* sum\_α g^{αμ} ∂\_α Φ  
 term\_cpl\_simplified = -psi \* sum(eta[alpha, mu]\*grad\_Phi[alpha] for alpha in range(4))  
 dL\_d\_dpsi.append(term\_kin + term\_cpl\_simplified)  
  
J\_scale = [sp.simplify(dL\_d\_dpsi[mu]\*psi) for mu in range(4)]  
div\_J\_scale = sp.simplify(sum(sp.diff(J\_scale[mu], coords[mu]) for mu in range(4)))  
  
print("Φ (Effective Potential) =", Phi)  
print("∇²Φ =", laplacian\_Phi)  
print("Lagrangian density L =", L)  
print("Noether current J^μ\_(ψ-scale) =", J\_scale)  
print("Flat divergence ∂\_μ J^μ\_(ψ-scale) =", div\_J\_scale)